Integration by Inverse Trigonometric Substitution

An integral which involves an integer power of x and an integer power of one of the square roots $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, or $\sqrt{x^2-a^2}$ can be transformed into an integral involving integer powers of trigonometric functions and no square roots. This is done by using an ARCSINE, ARCTANGENT, or ARCSECANT substitution.

1. Integrals involving integer powers of x and $\sqrt{a^2-x^2}$ where $a>0$

- Integrals of this type are transformed into integrals involving integer powers of sines and cosines by the substitution:
  - $\theta = \arcsin\left(\frac{x}{a}\right)$

  For which

  - $x = a\sin\theta$
  - $dx = \frac{d}{d\theta}[a\sin\theta]d\theta = a\cos\theta d\theta$
  - $\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$
  - See Figure 1

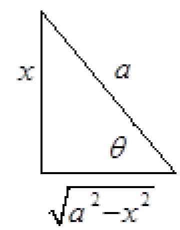

Figure 1

2. Integrals involving integer powers of x and $\sqrt{a^2+x^2}$ where $a>0$

- Integrals of this type are transformed into integrals involving integer powers of tangents and secants by the substitution:
  - $\theta = \arctan\left(\frac{x}{a}\right)$

  For which

  - $x = a\tan\theta$
  - $dx = \frac{d}{d\theta}[a\tan\theta]d\theta = a\sec^2\theta d\theta$
  - $\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$
  - See Figure 2

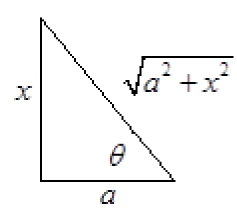

Figure 2

3. Integrals involving integer powers of x and $\sqrt{x^2-a^2}$ where $a > 0$

- Integrals of this type are transformed into integrals involving integer powers of secants and tangents by the substitution:
  - $\theta = \text{arcsec}\left(\dfrac{x}{a}\right)$

  For which

  - $x = a\sec\theta$
  - $dx = \dfrac{d}{d\theta}[a\sec\theta]d\theta = a\sec\theta\tan\theta d\theta$
  - $\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = a|\tan\theta|$
  - See Figure 3

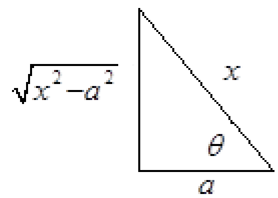

Figure 3