# Notes on Sequences

1. Definition of a sequence

   A sequence can be thought of as a real valued function whose domain is the set of positive integers. Expressed mathematically:

   $f(n) = a_n$ Where, n = a positive integer in the domain of the function $f$ and $a_n$ is the $n^{th}$ term of the sequence.

2. Limit of a sequence

   A sequence is said to have the limit L if we can make the absolute value of the difference between the terms of sequence and its limit as arbitrary small as we wish by taking $n$ sufficiently large. Expressed mathematically:

   $\lim_{n\to\infty} a_n = L$ if for each $\varepsilon > 0$, there exits $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$.

   If the limit $L$ exits, the sequence is said to converge to $L$. Otherwise, the sequence diverges.

3. Properties of Limits of Sequences

   Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$ then,

   - $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$
   - $\lim_{n\to\infty} ca_n = cL,$ $c$ is any real number
   - $\lim_{n\to\infty}(a_n b_n) = LK$
   - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K},$ $b_n \neq 0$ and $K \neq 0$

4. Theorem

   Let $L$ be a real number. Let $f$ be a function of a real variable such that

   $$\lim_{x\to\infty} f(x) = L$$

   If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer $n$, then

   $$\lim_{n\to\infty} a_n = L.$$

5. Absolute Value theorem for sequences

   For the sequence $\{a_n\}$, if
   $\lim_{n\to\infty}|a_n| = 0,$ then $\lim_{n\to\infty} a_n = 0$

6. Monotonic sequence

   A sequence is said to be monotonic if it's terms are either non-decreasing or non-increasing.

7. A sequence $\{a_n\}$ is said to be bounded above if there is a real number $U$ for which $a_n \le U$ for all n. $U$ is called an upper bound.

8. A sequence $\{a_n\}$ is said to be bounded below if there is a real number $L$ for which $a_n \ge L$ for all n. $L$ is called a lower bound.

9. A sequence $\{a_n\}$ is bounded if it is both bounded above and below.

10. By the completeness property of real numbers, if a sequence has an upper bound it has a least upper bound and if a sequence has a lower bound it has a greatest lower bound.

11. Theorem (Bounded monotonic sequences)

    If a sequence $\{a_n\}$ is both monotonic and bounded, it converges.